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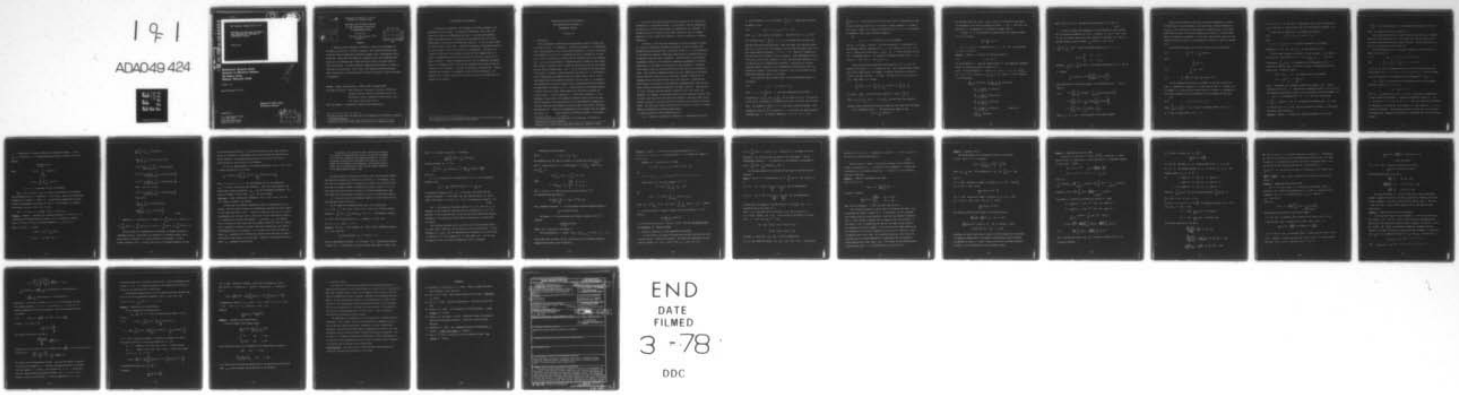

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ROBUSTNESS AND EFFICIENCY PROBLEMS OF SOME RANDOMIZATION PROCEDURES IN EXPERIMENTAL DESIGNS

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ABSTRACT

A concept of model-robustness is defined in terms of the performance of the design in the presence of model violations. The robustness problem is discussed for several randomization procedures commonly used in experimental design situations. Among them, the completely randomized design, the randomized block design and the randomized Latin square design are shown to be model-robust in their own settings. To compare different randomization procedures, a concept of efficiency is also defined. This concept, when applied to different designs, gives results which are consistent with the intuitive grounds on which the designs are suggested.

AMS(MOS) Subject Classifications: Primary 62K99, Secondary 62G35.

Key Words and Phrases: Model robustness, Minimaxity, Invariance, Efficiency, Systematic design, Completely randomized design, Randomized block design, Randomized cluster design, Crossover design, Latin square design.

Work Unit Number 4 - Probability, Statistics and Combinatorics.

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[1]The work was done while the author was in the Department of Statistics, University of California-Berkeley.

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SIGNIFICANCE AND EXPLANATION

In designing an experiment, the treatments are usually assigned to the experimental units randomly. In Planning of Experiments (Cox, 1958), two reasons for practicing randomization are given: to prevent the systematic biases from unknown sources of variation and to enable the error to be estimated whatever the form of the uncontrolled variation. So far all these reasons are stated in quite vague forms. A model robust approach is developed rigourously in this paper to justify the practices of randomization. The idea is to consider a collection of all possible violations of the assumed model and to choose a randomized design which will perform well over such a collection of model violations. Classical randomized designs like the completely randomized design, the randomized block design and the randomized Latin square design are justified in this framework. A concept of efficiency is also defined. Different randomization procedures are compared in terms of their efficiencies. Our approach thus provides a quantitative basis on which various randomization procedures can be assessed.

ROBUSTNESS AND EFFICIENCY PROBLEMS OF SOME RANDOMIZATION PROCEDURES IN EXPERIMENTAL DESIGNS

Chien-Fu Wu[1]

1. Introduction

The old problem of randomization in experimental design is treated from a new point of view. For comparing T different treatments on N experimental units, the experimenter, with an exact model assumption in his mind, will assign the treatments to the units in a systematic way which is optimal in the usual sense. This is certainly not a good ground for justifying the use of randomization. In fact, the experimenter's information about the model is never perfect. When a model is proposed, there is always the possibility that the "true" model deviates from the assumed model. Let G be the collection of all the possible "true" models. Concepts of model-robustness with respect to G are defined in terms of minimizing the maximum possible mean square errors (m.s.e.) of the corresponding best linear unbiased estimator (B.L.U.E.) over G. In Section 2, some randomization procedures commonly used in experimental design are shown to be model-robust with respect to any G which possesses an invariance property. A similar result for simple random sampling was obtained by Blackwell and Girshick (1954). For non-invariant G, a sufficient condition on G for minimaxity is also given. This optimality property actually holds for other types of problems, including violations of the homoscedasticity assumption of the error terms and the estimation of variance under either type of violation. Although many good properties of randomization are well-known (see, for example, Cox, 1958), nor[illegible]em are stated as an optimality property. The minimax

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s proved in this paper are thus new justifications for the practices of randomization in experimental design. As a by-product, several randomization procedures for the Latin square design are reassessed in this framework. The standard practice of choosing a Latin square arbitrarily and then randomly permuting its rows, columns and treatments is shown to possess some desirable properties.

In Section 3, the efficiency comparisons of several procedures (systematic, partially randomized, completely randomized) are made in terms of the maximum possible bias square over a particular choice of G. The calculations of the maxima are reduced to some simple combinatorics by considering the set of extreme points only (due to convexity). The efficiency of the systematic arrangements relative to the completely randomized arrangements is inversely proportional to the number of replications of each treatment. A randomized block design is shown to be very efficient when the block size is moderate or large. Other partially randomized designs like the randomized cluster design and the cross-over design are not as efficient, although a combined use of randomized block and cluster designs can be very efficient. Conditions for these designs to be superior to the completely randomized design (CRD) are also given in terms of the special patterns of the model violations. These patterns are quite consistent with the intuitive grounds on which these designs are suggested. For example, the cross-over design with two treatments is better than the CRD if more then 85% of the blocks have "positive" cross-over effect (to be defined in Section 3) and less than 15% of the blocks have "negative" cross-over effect. The efficiency concept introduced here thus makes it possible to compare different randomization procedures quantitatively.

2. Model-robustness of some randomized designs

Let T different treatments be compared on N experimental units with

n_t units assigned to the t-th treatment, $\sum_{t=1}^{T} n_t = N$. Assume that the following model is true:

$$(2.1) \qquad y_{ut} = \mu + \alpha_t + \varepsilon_{ut} \quad (u = 1, \cdots, N) \ ,$$

where y_{ut} is the response of treatment t assigned to unit u, α_t is the t-th treatment effect, $\sum_{t=1}^{T} n_t \alpha_t = 0$, ε_{ut} is the random error with zero mean and equal and uncorrelated variances. Under this model, any arbitrary assignment of n_t units to the t-th treatment gives the same mean square error of the best linear unbiased estimator $\hat{\alpha}_t$ $(t = 1, \cdots, T)$. Therefore, randomization does not seem to be necessary in such a situation. From a mathematical point of view, such an exact model assumption, in fact, any exact model assumption, can not justify the use of randomization procedures. Instead we will consider *different possible* violations of the assumed model and find designs which are minimax with respect to these model violations. This is the rationale of the model-robust approach adopted in this paper.

To include some other unknown effects, the following modified model is considered:

$$(2.2) \qquad y_{ut} = \mu + \alpha_t + g(u,t) + \varepsilon_{ut} \ ,$$

where $g \in G = \{g: \sum_{u=1}^{N} g(u,t) = 0$ and other assumptions to be defined$\}$,

assumptions on $\{\alpha_t\}_{t=1}^{T}$ and $\{\varepsilon_{ut}\}_{u=1}^{N}$ are the same as in (2.1). The function $g(u,t)$ can be interpreted as the unknown joint effect of the unit and the treatment. The assumption $\sum_{u=1}^{N} g(u,t) = 0$ is for obvious technical reasons. Other assumptions, which are not yet defined, on G will reflect the model-builder's knowledge about g. An important example of G is $\{g : |g| \leq c$ and

$\sum_{u=1}^{N} g(u,t) = 0\}$. This reflects the fact that very little is known about the type of model violation except that it is possible to happen anywhere in a neighborhood of zero. Model (2.2) without the specification of g in G was considered by Kempthorne (1955) from a different standpoint.

Let

$$I_t = \{u : u \text{ is assigned to the t-th treatment}\}$$

and call $I = \{I_t\}_{t=1}^{T}$ a pattern. In design terminology I corresponds to a non-randomized design. Mathematically I is just a partition of $\{1,\cdots, N\}$ into T subsets with cardinalities $\{n_t\}_{t=1}^{T}$. Let $\mathcal{I}$ be the collection of such I's. A randomized design η is defined as a probability measure over $\mathcal{I}$, i.e., $\{\eta(I)\}_{I\in\mathcal{I}}$ with $\eta(I) \geq 0$ and $\sum_{I\in\mathcal{I}} \eta(I) = 1$.

Since the statistician knows very little about the g's in (2.2), he might as well assume $g = 0$ in estimating $\{\alpha_t\}$. This is particularly suitable when no more known effects, for example, covariates, blocking etc. can be included in the model and when c is small. The sum of m.s.e.'s of the B.L.U.E. $\hat{\alpha}_t$'s, under model (2.2), becomes

$$\sum_{t=1}^{T} E\{(\hat{\alpha}_t - \alpha_t)^2\} = \sum_{t=1}^{T} \{\frac{1}{n_t} \sum_{u\in I_t} g(u,t)\}^2 + \sigma^2 \sum_{t=1}^{T} (\frac{1}{n_t} - \frac{1}{N}) \quad .$$

For fixed $\{n_t\}_{t=1}^{T}$, the only relevant quantity is $a(I,g) = \sum_{t=1}^{T} g_{.t}^2$,

where $g_{.t} = n_t^{-1} \sum_{u\in I_t} g(u,t)$. If $\{n_t\}_{t=1}^{T}$ are not fixed, they should be chosen as equal as possible and then bias reduction only needs to be considered.

For a randomized design η, the expected bias square is then

$$r(\eta,g) = \sum_{I\in\mathcal{I}} \eta(I)a(I,g) \quad .$$

This becomes a game with "risk" $r(\eta,g)$ between the experimenter and nature; the experimenter can choose any probability measure η over $\mathcal{I}$ and nature can choose any g, corresponding to an unknown true model, from G.

A design η^* is called minimax (or model-robust) with respect to model (2.2) if it achieves

$$\min_{\eta} \max_{g\in G} r(\eta,g) \quad .$$

For $g \in G$ and π a permutation of $\{1,\cdots, N\}$, let πg be defined as $\pi g(u,t) = g(\pi^{-1}u,t)$.

Theorem 1. Suppose G satisfies the invariance property:

$$g \in G \Rightarrow \pi g \in G \tag{2.3}$$

for any permutation π. Then the uniform design η^*, the completely randomized design, is minimax with respect to model (2.2).

Proof. For any permutation π, define $\pi I_t = \{\pi(u) : u \in I_t\}$, $\pi I = \{\pi I_t\}_{t=1}^{T}$ and $\eta_\pi(I) = \eta(\pi I)$. Given any η, $\frac{1}{N!}\sum_{\pi\in P} \eta_\pi = \eta^*$, where P is the collection of all permutations.

$$\max_{g\in G} \sum_{I\in g} \eta^*(I)a(I,g) = \frac{1}{N!} \max_{g\in G} \sum_{\pi\in P} \{ \sum_{I\in\mathcal{I}} \eta_\pi(I)a(I,g)\}$$

$$\leq \frac{1}{N!} \sum_{\pi\in P} \max_{g\in G}\{ \sum_{I\in\mathcal{I}} \eta(\pi I)a(I,g)\}$$

$$= \frac{1}{N!} \sum_{\pi\in P} \max_{g\in G}\{ \sum_{I\in\mathcal{I}} \eta(\pi I)a(\pi I,\pi g)\}$$

$$= \frac{1}{N!} \sum_{\pi\in P} \max_{g\in G}\{ \sum_{J\in\mathcal{I}} \eta(J)a(J,\pi g)\}$$

$$= \frac{1}{N!} \sum_{\pi\in P} \max_{g\in G}\{ \sum_{J\in\mathcal{I}} \eta(J)a(J,g)\} \qquad \text{(from (2.3))}$$

$$= \max_{g\in G} \sum_{J\in\mathcal{I}} \eta(J)a(J,g) \quad ,$$

where $a(\pi I, \pi g) = a(I,g)$ follows from the definition of πI and πg.

Q.E.D.

In order to compare the robustness of various designs in a more definite way, the following assumption on $g(u,t)$ is needed. If $g(u,t) = g(u) + h(t)$, i.e. no "interaction" between unit and treatment, by absorbing $h(t)$ into α_t, the only relevant quantity becomes $\sum_{I \in \mathcal{I}} \eta(I) a(I,g)$ with $a(I,g)$

$= \sum_{t=1}^{T} \{n_t^{-1} \sum_{u \in I_t} g(u)\}^2$. Define the random variables for $t = 1, \cdots, T$

$$x_t(u) = \begin{cases} 1 & \text{if} \quad u \in I_t \ , \\ 0 & \text{if} \quad u \notin I_t \ . \end{cases}$$

Therefore $\sum_{u \in I_t} g(u) = \sum_{u=1}^{N} g(u)\, X_u(t)$. If we further assume that $n_t = n$ for all t, we have

$$\sum_{I \in \mathcal{I}} \eta(I) a(I,g) = n^{-2} E\left[\sum_{t=1}^{T} \left\{ \sum_{u=1}^{N} g(u)\, X_t(u) \right\}^2 \right] \ ,$$

where expectation is taken with respect to the probability over $\mathcal{I}$. This is equal to

$$(2.4) \qquad n^{-2} E\left\{ \sum_{t=1}^{T} \sum_{u=1}^{N} g^2(u) X_t^2(u) + \sum_{t=1}^{T} \sum_{u \neq v} g(u) g(v) X_t(u) X_t(v) \right\}$$

$$= n^{-2} E\left[\sum_{u=1}^{N} g^2(u) + \sum_{u \neq v} g(u) g(v) \left\{ \sum_{t=1}^{T} X_t(u) X_t(v) \right\} \right]$$

$$= n^{-2} \left\{ \sum_{u=1}^{N} g^2(u) + \sum_{u \neq v} g(u) g(v)\, \pi_{uv} \right\} \ ,$$

where $\pi_{uv} = P$ (u and v being assigned to the same treatment).

Under the above additivity and equal replication assumptions, the degree of robustness of a design depends on its second order inclusion probabilities $\{\pi_{uv}\}_{u\neq v}$, which often have an interpretation in sampling theory. This simple device makes possible the efficiency comparisons of Section 3. Another advantage of the additivity assumption is in the Latin square design, where the less conclusive result of Theorem 3 is strengthened in Corollary 1 under this assumption.

The following relations and (2.4) are useful for later calculations.

$$\sum_{u=1}^{N} g^2(u) = -\sum_{u\neq v} g(u)g(v) \quad , \tag{2.5}$$

since

$$\sum_{u=1}^{N} g(u) = 0 \quad .$$

$$\sum_{\substack{v \\ v\neq u}} \pi_{uv} = (n-1) \quad . \tag{2.6}$$

$$\pi^{*}_{uv} = \frac{n-1}{N-1}, \text{ for equal } \pi^{*}_{uv}, \text{ where } N = nT \quad . \tag{2.7}$$

For the complete block designs, we assume, for the sake of simplicity, that T treatments are compared on b blocks each of size T. If model (2.8) is assumed, it is clear that the optimal design is to assign T treatments to each block irrespective of the arrangements of the treatments in the block.

Let

$$y_{ut} = \mu + \alpha_t + \beta_i + \varepsilon_{ut} \quad , \tag{2.8}$$

where $\sum_{t=1}^{T} \alpha_t = \sum_{i=1}^{b} \beta_i = 0$, α_t is the t-th treatment effect $(t=1,\cdots, T)$,

β_i is the i-th block effect $(i=1,\cdots, b)$,

$u = (i,j)$ $(j = 1,\cdots,T)$ are the T experimental units in the i-th block, and $\{\varepsilon_{ut}\}_u$ have mean 0, equal variance and are uncorrelated for different u's.

A more realistic modelling is to consider the following:

$$y_{ut} = \mu + \alpha_t + \beta_i + g(u,t) + \varepsilon_{ut} \quad , \tag{2.9}$$

where

$$g \in G = \{g : \sum_u g(u,t) = 0 \text{ and other assumptions to be defined}\} \quad ,$$

assumptions on $\{\alpha_t\}$, $\{\beta_i\}$ and $\{\varepsilon_{ut}\}$ are the same as in (2.8).

With the same rationale as in model (2.2) the estimate $\hat{\alpha}_t$ is chosen to be the B.L.U.E. under model (2.8), i.e. $\hat{\alpha}_t = \frac{1}{b} \sum_{u \in I_t} y_{ut} - y_{..}$. Since the corresponding $\sum_{t=1}^{T} \mathrm{Var}(\hat{\alpha}_t)$ is independent of the arrangements of the T treatments within each block, the choice of design is determined by the bias square terms. Formally we define

$$I_t^{(i)} = \{(i,j) : (i,j) \text{ is assigned to the t-th treatment}\}$$

$$I^{(i)} = \{I_t^{(i)}\}_{t=1}^{T} \quad , \quad I = \prod_{i=1}^{b} I^{(i)} \quad .$$

Each I corresponds to a systematic complete block design. Let $\mathcal{I}$ be the collection of all such I's. Definitions of randomized design η, $a(I,g)$, $r(\eta,g)$ and minimaxity with respect to model (2.9) are analogous to those for model (2.2).

Let π_i be a permutation of $\{1,\cdots, T\}$ for the i-th block and $\pi = \prod_{i=1}^{b} \pi_i$. Further πI and πg are defined in an obvious way. It is easy to see that $a(\pi I, \pi g) = a(I,g)$ and thus the proof of Theorem 2 is essentially the same as that of Theorem 1.

Theorem 2. *Suppose G satisfies the invariance property (2.3) for all*

$\pi = \prod_{i=1}^{b} \pi_i$, the uniform design η^*, the randomized block design in the usual sense, is minimax with respect to model (2.9).

Since there does not exist a simple transformation which maps a transformation set of Latin squares (L.S.) into another set, the application of the invariance technique used before is more restricted. The following result is thus not as conclusive as the previous two.

The following model for L.S. designs is proposed in the same spirit as models (2.2) and (2.9):

$$y_{ut} = \mu + \alpha_t + \beta_i + r_j + g(u,t) + \varepsilon_{ut} \tag{2.10}$$

where $\sum_{t=1}^{T} \alpha_t = \sum_{i=1}^{T} \beta_i = \sum_{j=1}^{T} r_j = 0$, $u = (i,j)$,

α_t, β_i, r_j are the t-th treatment effect, the i-th row effect and the j-th column effect $(i,j,\ t = 1,\cdots, T)$, $g \in G = \{g : \sum_{u} g(u,t) = 0$ and other assumptions to be defined$\}$; we also make the usual assumptions on $\{\varepsilon_{ut}\}_u$.

Let $\{I_i\}_{i=1}^{k}$ be the totality of transformation sets of Latin squares. Formally we define, for $i = 1,\cdots, k$,

$$I_i = \{\pi S_i \ ; \ \pi \in P\} \ ,$$

where S_i is a generating L.S. for I_i and P is the group of permutations of rows, columns and treatments. A natural correspondence between I_i and I_j is the map : $\pi S_i \leftrightarrow \pi S_j$ for all $\pi \in P$. Note that here we do not identify the identical L.S.'s among the $(T!)^3$ squares, i.e. every square in I_i is considered as "different". As we shall see later, this will save a lot of effort of randomization, compared to the practice of randomizing over the set of reduced squares.

A randomized L.S. design is defined as a probability measure η over $\bigcup_{i=1}^{k} I_i$. The measure η^* is called minimax with respect to model (2.10) if it achieves

$$\min_{\eta} \max_{g \in G} r(\eta, g) \quad ,$$

where

$$r(\eta, g) = \sum_{I \in \bigcup_{i=1}^{k} I_i} \eta(I) a(I, g) \quad ,$$

$$a(I, g) = \sum_{t=1}^{T} \left(\frac{1}{T} \sum_{u \in I_t} g(u,t) \right)^2 \quad ,$$

$$I_t = \{u : u \text{ is assigned to the t-th treatment}\} \quad .$$

Write $\eta(I) = \eta(I|I_i)c_i$ with c_i equal to the probability of choosing the transformation set I_i containing I and $\eta(I|I_i)$ equal to the conditional probability of choosing I within I_i; by applying the permutations of rows, columns and treatments, the choices of $\eta(I|I_i)$ can be reduced. To simplify the notation, we write $\eta(I|i)$ for $\eta(I|I_i)$.

For $\pi \in P$ and $g \in G$, define $\pi g(u,t) = g(\pi^{-1}(u,t))$.

Theorem 3. Suppose G satisfies the invariance property (2.3) for all $\pi \in P$, in obtaining the minimax designs for model (2.10), it suffices to consider those η^*'s with $\eta^*(I|I_i)$ the uniform measure on I_i for each i.

Proof: For each η, we have

$$\eta^*(I|i) = (T!)^{-3} \sum_{\tau \in P} \eta(\tau I|i)$$

for all $I \in I_i$ and $i=1,\cdots, k$.

$$\max_{g\in G} \sum_{i=1}^{k} c_i \sum_{I\in \mathcal{I}_i} \eta^*(I|i)a(I,g)$$

$$= \max_{g\in G} \sum_{i=1}^{k} \sum_{\pi\in P} c_i\, \eta^*(\pi S_i|i)a(\pi S_i,g)$$

$$= (T!)^{-3} \max_{g\in G} \sum_{\tau,\pi\in P} \sum_{i=1}^{k} c_i\, \eta(\tau\pi S_i|i)a(\pi S_i,g)$$

$$\le (T!)^{-3} \sum_{\tau\in P} \max_{g\in G} \left\{ \sum_{\pi\in P} \sum_{i=1}^{k} c_i\, \eta(\tau\pi S_i|i)a(\pi S_i,g) \right\}$$

$$= (T!)^{-3} \sum_{\tau\in P} \max_{g\in G} \left\{ \sum_{\lambda\in P} \sum_{i=1}^{k} c_i\, \eta(\lambda S_i|i)a(\lambda S_i,\tau g) \right\}$$

(from $a(\tau I,\tau g) = a(I,g)$)

$$= (T!)^{-3} \sum_{\tau\in P} \max_{g\in G} \left\{ \sum_{\lambda\in P} \sum_{i=1}^{k} c_i\, \eta(\lambda S_i|i)a(\lambda S_i,g) \right\}$$

(from (2.3))

$$= \max_{g\in G} \sum_{\lambda\in P} \sum_{i=1}^{k} c_i\, \eta(\lambda S_i|i)a(\lambda S_i,g)$$

$$= \max_{g\in G} \left\{ \sum_{i=1}^{k} c_i \sum_{I\in \mathcal{I}_i} \eta(I|i)a(I,g) \right\} \quad .$$

Q.E.D.

Examples of G satisfying (2.3) include $\{g : \sum_{u=1}^{N} g(u,t) = 0,\ |g(u,t)| \le c\}$, $\{g : \sum_{u=1}^{N} g(u,t) = 0,\ \sum_{u=1}^{N} |g(u,t)| \le c\}$ and $\{g : \sum_{u=1}^{N} g(u,t) = 0,\ \sum_{u=1}^{N} g^2(u,t) \le c\}$.

Several reasons for randomization are mentioned in standard textbooks on experimental design. Among them, the "prevention of systematic biases from unknown sources of variation" is closely related to our Theorems 1, 2 and 3. Another important reason - to enable the error to be estimated whatever the form

of the uncontrolled variation - is also closely related to the result stated at the end of the section. Unfortunately none of these reasons are stated as optimality properties. The minimax results proved in this section are thus new justifications for the practices or randomization.

For the L.S. design problem, if we further assume that $g(u,t) = g(u) + h(t)$, a similar calculation to (2.4) gives

$$\sum_{I \in \mathcal{I}} \eta(I)a(I,g) = \frac{1}{T^2}\left(\sum_{i,j=1}^{T} g^2(i,j) + \sum_{\substack{i \neq i' \\ j \neq j'}} g(i,j)g(i',j')\pi_{uu'} \right) ,$$

where $u = (i,j)$, $u' = (i',j')$ are two different units in the square and $\pi_{uu'}$ = P (u and u' receive the same treatment). Since this only depends on the second order inclusion probability $\pi_{uu'}$, all the randomization procedures η*'s stated in Theorem 3, having equal $\pi_{uu'}$'s, are thus minimax.

Corollary 1. Under the additional assumption $g(u,t) = g(u) + h(t)$, all the η*'s stated in Theorem 3 are minimax.

In particular, the complete randomization within a transformation set is minimax. This will greatly simplify the randomization procedure for Latin squares in the Fisher-Yates Tables (1953). Instead of randomizing first over the class of transformation sets and then within a particular set (the Fisher-Yates "recipe"), it is sufficient to consider any procedure η with equal π_{uv}. The simplest one is to choose any Latin square and then randomly permute its rows (or columns). This is especially convenient for higher order Latin squares where the class of transformation sets is not available. It is interesting to note that both the equal $\pi_{uu'}$ procedure and the Fisher-Yates procedure were mentioned in Fisher's definition of Latin square in his 1926 paper. However, no justification for the equal $\pi_{uu'}$ procedure was given there:

Consequently, the term Latin Square should only be applied to a process of randomization by which one is selected at random out of the total number of Latin Squares possible; or, at least, to specify the agricultural requirement more strictly, out of a number of Latin Squares in the aggregate, of which every pair of plots, not in the same row or column, belongs equally frequently to the same treatment (Fisher, 1926).

The same invariance technique can be used to show that some other randomized designs are minimax with respect to an invariant G. For example, in the BIBD case, the standard method of randomizing the blocks, the units within each block and the treatment numbers is minimax with respect to such a G; in the first order multi-factor design, the device of "angular randomization" (Box, 1952) makes the design minimax with respect to the set of second order models.

If G does not possess the invariance structure, to what extent will the above minimax results still hold?

For the completely randomized design (CRD), let us assume that $n_t = n$ $(t=1,\cdots, T)$ and $g(u,t) = g(u) + h(t)$. From (2.4), the only relevant quantity is $\sum_{u=1}^{N} g^2(u) + \sum_{u\neq v} g(u)g(v)\pi_{uv}$ for $g \in G$. Its maximum is attained at $\mathcal{E}$, the set of extreme points of G. Decompose $\mathcal{E} = \cup_r Ar$ with $Ar = \{g : g \in G , \|g\|^2 = r\} \cap \mathcal{E}$.

Theorem 4. For any r with non-empty Ar, there exists a probability measure p_r on Ar such that

$$\int_{Ar} g(u)g(v)p_r(dg) = c_1 < 0 \quad \text{for all} \quad u \neq v \ .$$

Then any randomization procedure η^* with equal π^*_{uv} is minimax with respect to model (2.2). In particular, the CRD is minimax with respect to model (2.2).

Proof. It is enough to show that η^* minimizes

$$\max_{g \in Ar} \{ \sum_{u=1}^{N} g^2(u) + \sum_{u \neq v} g(u)g(v)\pi_{uv} \}$$

for each non-empty Ar. On Ar,

$$\sum_{u=1}^{N} g^2(u) + \sum_{u \neq v} g(u)g(v)\pi_{uv} = r + \frac{n-1}{N-1} \sum_{u \neq v} g(u)g(v) = \frac{N-n}{N-1} r \quad ,$$

from (2.5).

If p_r is considered as a "Bayesian prior" on Ar, the corresponding "Bayesian risk"

$$\int_{Ar} \{r + \sum_{u \neq v} \pi_{uv} g(u)g(v)\} p_r(dg) = r + c_1 \sum_{u \neq v} \pi_{uv}$$

is minimized by taking all the π_{uv}'s to be equal. This shows that the "decision procedure" η^* with equal π^*_{uv} has constant risk on Ar and is "Bayes" with respect to a "prior" p_r; thus it is minimax with respect to Ar.

Q.E.D.

For G satisfying the invariance property (2.3), one choice of p_r in Theorem 4 is any probability invariant under the permutations of $\{1, \cdots, N\}$. Theorem 4 is thus a generalization of the previous results under some stronger assumptions. Its extensions to other design situations are straightforward but tedious.

The technique developed so far can also be used to give minimax results for other types of problems. Two of these that will be discussed are: (i) the estimation of $\{\alpha_t\}_{t=1}^{T}$ when the homoscedasticity assumption of the error terms is violated and (ii) the estimation of σ^2 under model (2.2). For the sake of simplicity, only the complete randomization case is considered.

Consider the following model

(2.11) $$y_{ut} = \mu + \alpha_t + \varepsilon_{ut} \quad .$$

The assumptions are the same as in model (2.1) except that $\mathrm{Cov}(\{\varepsilon_{ut}\}_{u=1}^{N}) = \Sigma$ with Σ coming from a set E. For any pattern $I = \{I_t\}_{t=1}^{T}$, the B.L.U.E. of $\{\alpha_t\}_{t=1}^{T}$ is

$$\{\hat{\alpha}_t\}_{t=1}^{T} = \{y_{t\cdot} - y_{\cdot\cdot}\}_{t=1}^{T} = (P - J)\underset{\sim}{y} \quad ,$$

where

$$P = [a_{tj}]_{t\times N}, \quad a_{tj} = \begin{cases} \frac{1}{n_t} & \text{if } j \in I_t \ , \\ 0 & \text{if } j \notin I_t \ , \end{cases}$$

and J is the $T \times N$ matrix with all the entries equal to N^{-1}.

Its variance-covariance matrix is

$$\mathrm{Cov}\left(\{\hat{\alpha}_t\}_{t=1}^{T}\right) = (P - J)\Sigma(P - J)^T \triangleq A(I,\Sigma) \quad .$$

For a randomized procedure η, the corresponding variance-covariance matrix is

$$\sum_{I\in\mathcal{I}} \eta(I)A(I,\Sigma) \triangleq R(\eta,\Sigma) \quad .$$

The design η^* is called minimax with respect to model (2.11) if it achieves

$$\min_{\eta} \max_{\Sigma\in E} \mathrm{tr}\{R(\eta,\Sigma)\} \quad ,$$

where $\mathrm{tr}R$ is the trace of the matrix R.

For any permutation π, define $(\pi\Sigma)_{ij} = (\Sigma)_{\pi^{-1}i,\pi^{-1}j}$ for all i,j. It is easy to see that $A(\pi I,\pi\Sigma) = A(I,\Sigma)$ and the proof of the following theorem is essentially the same as that of Theorem 1.

Theorem 5. Suppose E satisfies the invariance property (2.3), i.e. $\Sigma \in E \Rightarrow \pi\Sigma \in E$ for all π, the uniform design η^* is minimax with respect to model (2.11).

Examples of E satisfying (2.3) include

$$E_0 = \{\Sigma = [\sigma_{ij}]_{ij} : \sigma_{ij} \in (a,b) \text{ for all } i,j\}$$

and

$$E_1 = \{\Sigma = [\sigma_{ij}]_{i,j} : \sigma_{ii} = \sigma^2, \sigma_{ij} = \rho\sigma^2,\ 0 < \sigma^2 < \infty,\ -1 \le \rho \le 1\} .$$

Under model (2.2), the usual estimator of σ^2 is

$$\hat{\sigma}^2 = (N-T)^{-1} \sum_{t=1}^{T} \sum_{u \in I_t} (y_{ut} - y_{\cdot t})^2$$

and

$$E(\hat{\sigma}^2) - \sigma^2 = (N-T)^{-1} \sum_{t=1}^{T} \sum_{u \in I_t} (g(u,t) - g_{\cdot t})^2 ,$$

where $g_{\cdot t} = n_t^{-1} \sum_{u \in I_t} g(u,t)$. Let $a(I,g) = \sum_{t=1}^{T} \sum_{u \in I_t} (g(u,t) - g_{\cdot t})^2$. A design η^* is called minimax with respect to model (2.2) for estimating σ^2 if it achieves

$$\min_{\eta} \max_{g \in G} \sum_{I \in \mathcal{I}} \eta(I) a(I,g) .$$

It is clear that the relation $a(\pi I, \pi g) = a(I,g)$ holds and the minimax result for estimating σ^2 follows as usual.

3. Efficiency comparisons of some randomization procedures.

In this section, we attempt to evaluate more precisely the gains and losses in using various randomization procedures. The efficiency comparisons are made under the assumptions: $g(u,t) = g(u) + h(t)$, $n_t = n$ $(t=1,\cdots,T)$ and

$G = \{g : \sum_{u=1}^{N} g(u,t) = 0, |g(u,t)| \leq c\}$. Both model (2.2) and model (2.10) are considered. All the calculations are based on (2.4) and Lemma 1. For any randomization procedure η, its efficiency is then measured by the maximum of $s(\eta,g) = \sum_{u=1}^{N} g^2(u) + \sum_{u \neq v} g(u)g(v)\pi_{uv}$ over G.

The following lemma will be referred to very often in the efficiency calculations.

Lemma 1. The set E of extreme points of $\{g : \sum_{u=1}^{N} g(u) = 0$ and $|g(u)| \leq c\}$ is

(3.1) (i) For N even, $\{c(\underbrace{1,\cdots,1}_{\frac{N}{2}}, \underbrace{-1,\cdots,-1}_{\frac{N}{2}})$ and its permutations$\}$

(3.2)(ii) For N odd, $\{c(0, \underbrace{1,\cdots,1}_{\frac{N-1}{2}}, \underbrace{-1,\cdots,-1}_{\frac{N-1}{2}})$ and its permutations$\}$.

In particular, the maximum of a convex function of $\{g(u)\}_{u=1}^{N}$ over G is attained at one of the points in E.

Proof. It is clear that the set of points in (3.1) and (3.2) are in E.

(i) N even. Suppose $c(a_1,\cdots,a_N)$ is not a point of the form (3.1), we can find $i < j$ with $|a_i|, |a_j| < 1$. Also

$$(a_1,\cdots,a_N) = \frac{1}{2}(a_1,\cdots,a_i-\delta,\cdots,a_j+\delta,\cdots,a_N) +$$

$$\frac{1}{2}(a_1,\cdots,a_i+\delta,\cdots,a_j-\delta,\cdots,a_N)$$

for small δ shows that $(a_1,\cdots,a_N)$ is not an extreme point.

(ii) N odd. Among the points $(a_1,\cdots,a_N)$ with $|a_i| = 0$ or 1, only the set

of points in (3.2) are in E. Otherwise, we can find $i < j$ with $|a_i|, |a_j| < 1$. The rest of the proof follows from (i).

Q.E.D.

From the result of Lemma 1, the relative efficiency of one procedure to another is independent of c, the radius of the neighborhood G. Without loss of generality we can therefore assume that c is equal to 1 in the following efficiency comparisons.

Example 1. Completely Randomized Design (CRD).

From (2.5) and (2.7),

$$s(\eta,g) = (1 - \frac{n-1}{N-1}) \sum_{u=1}^{N} g^2(u) \quad .$$

According to Lemma 1,

$$\max_{g \in G} s(\eta,g) = \begin{cases} \frac{N-n}{N-1} N & \text{for } N \text{ even} \\ N-n & \text{for } N \text{ odd} \end{cases} .$$

Note that the maximizing g can be any point from the set E.

This calculation amounts to saying that all procedures with equal π_{uv}'s are minimax with respect to model (2.2). In particular, the CRD is minimax. The following question arises naturally: does there exist another minimax design η whose support is smaller than the support of the CRD, the whole I? The answer is yes! By identifying the CRD with the simple random sampling, the radomized block design (example 3) with the stratified random sampling and the randomized cluster design (example 4) with the cluster sampling, a result in sampling theory can be used. In the design terminology, this result says that there exists a convex combination of randomized block and randomized cluster designs which gives equal π_{uv}'s. The support of this combined design is much smaller than I. For technical details, see Wynn (1977).

Example 2. Systematic Design.

The design measure for the systematic design is a point mass $\eta_{\tilde{I}}$ with $\eta_{\tilde{I}}(\tilde{I}) = 1$, $\tilde{I}$ is a pattern.

$$s(\eta_{\tilde{I}}, g) = \sum_{t=1}^{T} \left(\sum_{u \in I_t} g(u) \right)^2 = \sum_{t=1}^{T} A_t^2$$

where $A_t = \sum_{u \in I_t} g(u)$. The constraints on $\{A_t\}$ are $\sum_{t=1}^{T} A_t = 0$ and $|A_t| \leq n$.

(i) T even. According to Lemma 1, by taking $A_t = n$ for t even and $A_t = -n$ for t odd, we have

$$\max_{g \in G} s(\eta_{\tilde{I}}, g) = T n^2 = Nn$$

(ii) T odd. According to Lemma 1, by taking $A_t = n$ for t even, $A_t = -n$ for t odd ($\leq T-1$) and $A_T = 0$, we have

$$\max_{g \in G} s(\eta_{\tilde{I}}, g) = (T-1)n^2 \quad .$$

The relative efficiency with respect to the minimax value is

$$\frac{N-n}{N-1} N/Nn = \frac{N-n}{N-1} /n < \frac{1}{n} \quad \text{for } T \text{ even},$$

$$\frac{N-n}{N-1} N/(T-1)n^2 = T/(nT-1) \simeq \frac{1}{n} \quad \text{for } T \text{ odd and } n \text{ even},$$

$$N-n/(T-1)n^2 = \frac{1}{n} \quad \text{for } T, n \text{ odd}.$$

Therefore the loss of efficiency in terms of the bias squares for the systematic design is proportional to the number of replications of each treatment. For moderate or large n, unless a specific pattern of the model violations is known, it is not advisable to use a systematic design.

Example 3. Randomized Block Design (RBD).

Divide the N units into ℓ blocks $\left\{I^{(i)}\right\}_{i=1}^{\ell}$ and assign T treatments each with $\frac{n}{\ell}$ replications (n being divisible by ℓ) completely randomly to each of the ℓ blocks.

$$\text{For } u \neq v \in I^{(i)}, \quad \pi_{uv} = T\,\frac{n(n-\ell)}{N(N-\ell)} = \frac{n-\ell}{N-\ell} .$$

$$\text{For } u \in I^{(i)}, v \in I^{(j)}, i \neq j, \quad \pi_{uv} = \frac{1}{T} .$$

From (2.5),

$$\sum_{u\neq v} g(u)g(v)\pi_{uv} = \frac{n-\ell}{N-\ell} \sum_{u\neq v\in I^{(i)}} g(u)g(v) + \frac{1}{T}\Big(-\sum_{u=1}^{N} g^2(u) - \sum_{u\neq v\in I^{(i)}} g(u)g(v)\Big) .$$

Therefore $$s(\eta,g) = \frac{T-1}{T}\sum_{u=1}^{N} g^2(u) - \frac{(N-n)\ell}{N(N-\ell)} \sum_{u\neq v\in I^{(i)}} g(u)g(v) .$$

From Lemma 1, it suffices to consider only the set E. Define

m_i = number of u from $I^{(i)}$ with $g(u)$ equal to 1,

n_i = number of u from $I^{(i)}$ with $g(u)$ equal to -1,

$$\sum_{u\neq v\in I^{(i)}} g(u)g(v) = \sum_{i=1}^{\ell} \{(m_i-n_i)^2 - (m_i+n_i)\} ,$$

$$s(\eta,g) = \left\{\frac{T-1}{T} + \frac{(N-n)\ell}{N(N-\ell)}\right\} \sum_{i=1}^{\ell} (m_i+n_i) - \frac{(N-n)\ell}{N(N-\ell)} \sum_{i=1}^{\ell} (m_i-n_i)^2$$

$$= \frac{N-n}{N-\ell} \sum_{i=1}^{\ell} (m_i+n_i) - \frac{\ell(N-n)}{N(N-\ell)} \sum_{i=1}^{\ell} (m_i-n_i)^2 . \tag{3.3}$$

This is maximized by taking $|m_i - n_i|$ as small as possible and $m_i + n_i$ as large as possible.

(i) $\frac{N}{\ell}$ even. By taking $m_i = n_i = \frac{1}{2}\frac{N}{\ell}$,

$$\max_{g\in G} s(\eta,g) = N\,\frac{N-n}{N-\ell} \quad .$$

(ii) $\frac{N}{\ell}$ odd. Obviously $m_i - n_i$ should be taken to be ± 1 or 0; for $m_i - n_i = \pm 1$, the biggest possible $m_i + n_i$ is $\frac{N}{\ell}$; for $m_i - n_i = 0$, the biggest possible $m_i + n_i$ is $\frac{N}{\ell} - 1$.

Let p = number of i's with $m_i - n_i$ equal to 1

= number of i's with $m_i - n_i$ equal to -1 ,

$\ell - 2p$ = number of i's with $m_i - n_i$ equal to 0 .

Therefore, $\sum_{i=1}^{\ell} (m_i - n_i)^2 = 2p$ and the biggest possible $\sum_{i=1}^{\ell} (m_i + n_i)$ is

$\frac{N}{\ell}\,2p + (\frac{N}{\ell} - 1)\,(\ell - 2p) = N - \ell + 2p$.

For such a g, $s(\eta,g) = N - n + 2p\,\frac{N-n}{N}$ is maximized by taking $2p = \ell$ for ℓ even and $\ell - 1$ for ℓ odd.

(I) ℓ even. $\max_{g\in G} s(\eta,g) = (N-n)(N+\ell)/N$.

(II) ℓ odd. $\max_{G\in G} s(\eta,g) = (N-n)(N+\ell-1)/N$.

The relative efficiency to the CRD is

$$\frac{\frac{N}{N-1}\,(N-n)}{N\,\frac{(N-n)}{N-\ell}} = \frac{N-\ell}{N-1} \quad \text{for } \frac{N}{\ell} \text{ even}$$

$$\frac{\frac{N}{N-1}\,(N-n)}{\frac{N+\ell}{N}\,(N-n)} = \frac{N^2}{(N+\ell)(N-1)} \quad \text{for } \frac{N}{\ell} \text{ odd, } \ell \text{ even}$$

$$\frac{N-n}{\frac{N+\ell-1}{N}\,(N-n)} = \frac{N}{N+\ell-1} \quad \text{for } \frac{N}{\ell} \text{ odd, } \ell \text{ odd.}$$

When $\frac{\ell}{N}$ is close to 0, the relative efficiency is close to 1. For moderate $\frac{\ell}{N}$, the loss of efficiency for the randomized block design is still very small. In general, if the block effect accounts for most of the unknown effect $g(u)$, the RBD will be more efficient than the CRD. More precisely, if $\sum_{i=1}^{\ell} (m_i - n_i)^2 > N$, the s-value of the RBD is, according to (3.3), smaller than $\frac{(N-n)N}{N-\ell} - \frac{\ell(N-n)}{N(N-\ell)} N = (N-n)$ which is smaller than or equal to the s-value of the CRD.

Example 4. Randomized Cluster Design (RCD).

Let $N = nT = pqT$, where p is the size of the cluster. Divide N units into qT clusters and randomly assign q of the qt clusters to each treatment. For u, v in the same cluster, $\pi_{uv} = 1$; otherwise $\pi_{uv} = (q-1)/(qT-1)$.

From (2.5),

$$s(\eta,g) = (1 - \frac{q-1}{qT-1}) \left\{ \sum_{u=1}^{N} g^2(u) + \sum_{\substack{u \neq v \\ \text{same cluster}}} g(u)g(v) \right\} .$$

The following calculations are justified by Lemma 1.

(i) qT even. By choosing $\frac{1}{2}qT$ clusters with all their $g(u)$'s to be 1 and the other $\frac{1}{2}qT$ clusters with all their $g(u)$'s to be -1,

$$\max_g s(\eta,g) = \frac{q(T-1)}{qT-1} (N + qTp(p-1)) = N \frac{N-n}{qT-1} .$$

(ii) qT odd, p even. By choosing $\frac{1}{2}(qT-1)$ clusters with all their $g(u)$'s to be 1, $\frac{1}{2}(qT-1)$ clusters with all their $g(u)$'s to be -1 and one cluster with half $g(u)$'s to be 1 and half $g(u)$'s to be -1,

$$\max_{g\in G} s(\eta,g) = \frac{q(T-1)}{qT-1}\,(N + (qT-1)p(p-1) - p)$$

$$\approx (N-p)\,(N-n)/(qT-1) \quad .$$

(iii) qT, p odd. A similar calculation to (ii) gives

$$\max_{g\in G} s(\eta,g) = (N-p)\,(N-n)/(qT-1) \quad .$$

The relative efficiency to the CRD is

$$\frac{qT-1}{N-1} < \frac{1}{p} \qquad \text{for } qT \text{ even} \ ,$$

$$\frac{N(qT-1)}{(N-p)(N-1)} \simeq \frac{1}{p} \qquad \text{for } qT \text{ odd, } p \text{ even} \ ,$$

$$\frac{qT-1}{N-p} = \frac{1}{p} \qquad \text{for } qT, p \text{ odd} \ .$$

For large or moderate p, the use of RCD alone is quite inefficient. However, combining the RCD with the RBD can improve the efficiency as was discussed in Example 2. The RCD is better than the RBD for small p and large ℓ and vice versa. A more precise comparison can be made based on their efficiencies.

Example 5. Cross-over Design with two treatments.

In a cross-over design with two treatments, N units are divided into $\frac{1}{2}N$ blocks $(1,2),\cdots,(N-1,N)$, with one unit receiving a treatment and the other unit in the same block receiving another treatment, the control. Of the $\frac{1}{2}N$ blocks, $\frac{1}{4}N$ blocks are selected at random for "treatment and then control"; the other $\frac{N}{4}$ blocks for "control and then treatment". To calculate the π_{uv}'s, it is sufficient to work out π_{13} and π_{14}, since for all i,

$$\pi_{2i+j,2i'+j'} = \pi_{13} \quad \text{for } j = j' = 1 \text{ or } 2$$

and $\pi_{2i+j,2i'+j'} = \pi_{14}$ for $j = 1$, $j' = 2$ or $j = 2$, $j' = 1$.

$$\pi_{13} = 2\binom{\frac{N}{2} - 2}{\frac{N}{4}} \Big/ \binom{\frac{N}{2}}{\frac{N}{4}} = \frac{N-4}{2(N-2)} = 1 - \pi_{14} \quad .$$

$$\sum_{u \neq v} g(u)g(v)\pi_{uv} = \frac{N-4}{2(N-2)} \sum_{i \neq i'} \{g(2i+1)g(2i'+1) + g(2i+2)g(2i'+2)\}$$

$$+ \frac{N}{2(N-2)} \sum_{i \neq i'} \{g(2i+1)g(2i'+2) + g(2i+2)g(2i'+1)\} \quad .$$

From Lemma 1, we need only consider $g(u) = \pm 1$. For each block, there are four possible patterns: ++,--,+-,-+. Let there be ℓ, ℓ, m, $\frac{N}{2} - 2\ell - m$ blocks corresponding to these four patterns. After some calculations which are omitted,

$$(3.4) \qquad s(\eta,g) = N - \frac{4}{N-2}\left(\frac{N}{2} - 2m - 2\ell\right)^2 + (-4\ell + 2)\frac{N}{N-2} \quad .$$

By taking $\ell = 0$ and $m = \frac{N}{4}$,

$$\max_{g \in G} s(\eta,g) = \frac{N^2}{N-2} \quad .$$

The relative efficiency to the CRD is

$$\frac{\frac{N}{N-1}(N - \frac{1}{2}N)}{\frac{N^2}{N-2}} = \frac{N-2}{2(N-1)} \leq \frac{1}{2} \quad .$$

Actually the $s(\eta,g)$ value in (3.4) is smaller than $\frac{N^2}{2(N-1)}$, the s-value of the CRD, if and only if $\left(\frac{\frac{1}{2}N - 2\ell - 2m}{\frac{1}{2}N}\right)^2 + \frac{\ell}{\frac{1}{2}N} > \frac{N}{2(N-1)} \simeq \frac{1}{2}$.

This result can be interpreted as follows: the cross-over design is superior to the CRD if the number of + - blocks is substantially larger (or smaller) than the number of - + blocks. The existence of ++ or -- blocks will make the cross-over design even more efficient. For $\ell = 0$ (no ++ or -- blocks), if 85% of the blocks are +- and the remaining 15% - +, the

cross-over design ties in efficiency with the CRD. The most advantageous case of 100% + - (or - +) blocks corresponds to the assumption that cross-over effects of the same kind exist in every block.

The efficiency comparisons of the Latin square design will be made under model (2.10) and the additional assumption $g(u,t) = g(u) + h(t)$ and

$$G = \{g : |g| \le 1, \sum_{i,j=1}^{T} g(u) = 0 \ , \ u = (i,j)\} \ .$$

Example 6. Randomized Latin Square Design.

For the randomized Latin square design,

$\pi_{uv} = \frac{1}{T-1}$ for u, v not in the same row and column. By (2.5), we have

$$(3.5) \qquad s(\eta,g) = \sum_{i,j=1}^{T} g^2(i,j) + \frac{1}{T-1} \sum_{\substack{i\neq i' \\ j\neq j'}} g(i,j)g(i',j')$$

$$= (1 - \frac{1}{T-1}) \sum_{i,j=1}^{T} g^2(i,j) - \frac{1}{T-1}\left\{ \sum_{\substack{j\neq j' \\ i}} g(i,j)g(i,j') + \sum_{\substack{i\neq i' \\ j}} g(i,j)g(i',j) \right\} \ .$$

(i) T even. According to Lemma 1, it suffices to consider the g(u)'s with half of them to be 1 and half of them to be -1. Let

m_i = number of u's with g(u) to be 1 in the i-th row,

m'_j = number of u's with g(u) to be 1 in the j-th column,

$n_i = T - m_i$, $n'_j = T - m'_j$.

$$s(\eta,g) = \frac{T-2}{T-1} T^2 - \frac{1}{T-1}\left[\sum_{i=1}^{T} \{(m_i - n_i)^2 - T\} + \sum_{j=1}^{T} \{(m'_j - n'_j)^2 - T\} \right]$$

is maximized by taking $m_i = n_i = \frac{1}{2}T$.

Therefore

$$\max_{g\in G} S(\eta,g) = \frac{T^3}{T-1} \ .$$

(ii) T odd. According to Lemma 1, it suffices to consider the $g(u)$'s with $\frac{1}{2}(T^2-1)$ of them to be 1, $\frac{1}{2}(T^2-1)$ of them to be -1 and one to be 0.

$$s(\eta,g) = \frac{T-2}{T-1}(T^2-1) - \frac{1}{T-1}\left\{\sum_{i=1}^{T}(m_i-n_i)^2 - T^2 + \sum_{j=1}^{T}(m'_j-n'_j)^2 - T^2\right\}$$

is maximized by taking $m_i - n_i$ to be 1 for i even ($i \leq T-1$), to be -1 for i odd ($i \leq T-1$) and to be 0 for $i = T$.

Therefore

$$\max_{g\in G} s(\eta,g) = \frac{T^3 - 3T + 4}{T-1} .$$

Example 7. Systematic Latin Square Design.

For the systematic Latin square design,

$$\max_{g\in G} s(\eta,g) = \max_{g\in G} \sum_{t=1}^{T}\{\sum_{u\in I_t} g(u)\}^2$$

$$= \begin{cases} T^3 & \text{for } T \text{ even}, \\ (T-1)T^2 & \text{for } T \text{ odd}. \end{cases}$$

The relative efficiency to the randomized Latin square design is equal to

$$\frac{1}{T-1} \quad \text{for } T \text{ even},$$

$$\frac{T^3 - 3T + 4}{T^3 - T^2}\frac{1}{T-1} \quad \text{for } T \text{ odd}.$$

It is also worth noting that the permutations of treatments only will give the same π_{uv} as the systematic design and hence is not advisable.

4. Concluding remarks.

One reason for incorporating randomization in experimental design is to spread out more evenly the risks incurred with the violations of the assumed model. This idea is formalized more clearly in our approach which consists of the concepts of neighborhood of model violations, invariance, design measure and minimaxity. But for designs with covariates, this approach does not seem to work that well. The corresponding minimax design measure depends on the configuration of the covariates and is in general very hard to obtain. Note that here the invariance technique employed in Section 2 fails. This is certainly a very interesting problem for future research.

One advantage of our approach lies in the possibility of making efficiency comparisons. Due to Lemma 1 and some other combinatorial arguments, it provides a basis for making quantitative assessments of various randomization procedures. The conclusions drawn from the computations in Section 3 are quite consistent with the intuitive grounds on which the designs are suggested. Such an idea, i.e. to reduce the efficiency calculations to simple combinatorics on the vertices of the neighborhood, may be of value in evaluating other randomization methods used in statistics and related fields.

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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Model robustness, Minimaxity, Invariance, Efficiency, Systematic design, Completely randomized design, Randomized block design, Randomized cluster design, Cross-over design, Latin square design.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A concept of model-robustness is defined in terms of the performance of the design in the presense of model violations. The robustness problem is discussed for several randomization procedures commonly used in experimental design situations. Among them the completely randomized design, the randomized block design and the randomized Latin square design are shown to be model-robust in their own settings. To compare different randomization procedures, a concept of efficiency is also defined. This concept, when applied to different designs, gives results which are consistent with the intuitive grounds on which the designs are suggested.

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